Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect. *Classical subjects*[1]

Vladimir Nodelman
nodelman@hit.ac.il

Department of Computer Sciences,
Holon Institute of Technology
52 Golomb St., Holon, 58102,
ISRAEL

Abstract

There exists a rather large amount of frequently eclectic assemblages [1]-[7] *of interesting, useful and famous counterexamples. Often discovered by great mathematicians, they played an important role in the history of mathematics. A proper software can make them and related learning activities more understandable, visual and user-friendly. We consider three subjects that have a certain connection, and include famous counterexamples, named after their discoverers.*

1. Length of curve

The following counterexamples motivate a correct definition of the notion *length of curve*.

1.1. Segment. The Amount of wrinkles in model **M1** (Fig.1) depends on parameter *k*. The blue and red polygonal chains[2] present two sequential steps of wrinkling. Fig.1 illustrates:

1. The length of these chains does not depend on *k* and remains equal to $\sqrt{2}$.
2. Both polygonal chains converge to the unit segment.

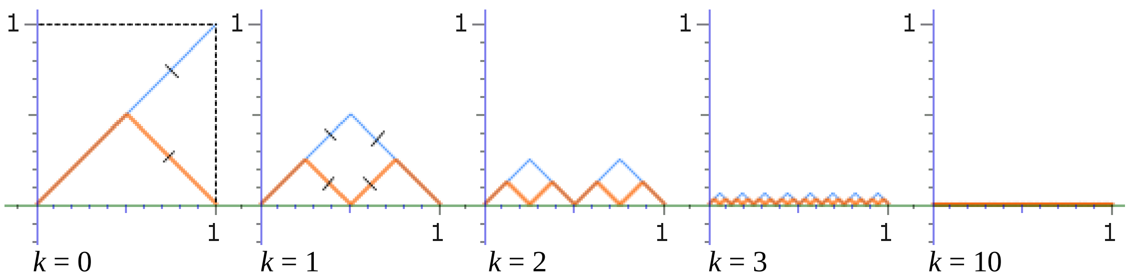


Figure 1

Thus, the length of unit segment is not 1 but $\sqrt{2}$!

Exercise: "*Prove*" that length of unit segment "*equals*" to any $L \geq 1$.

[1] It is the third part in the series of papers "*Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect*".

[2] Defined by syntaxic construct *polyline*(*X, Y, Z, index, amount, closed*) as

polyline(i*1/2^k,if(i%2,1/2^k,0),0,i,2^k+1,0) –blue, and

polyline(i*1/2^(k+1),if(i%2,1/2^(k+1),0),0,i,2^(k+1)+1,0) – red.

1.2. Circle. The Rolling Bridge at Paddington Basin in London[3] with its triangular sections impressed us to model it with *VisuMatica*.

This model **M2** (Fig.2) consists of congruent isosceles right triangles and depends on two parameters: k – the amount of triangles, and $t \in [0, 1]$ – the folding stage.

Students easily find the analogy with the case of polygonal chains converging to segment in 1.1. However, there is a visible difference:

a) The length of the chain projection onto the x-axis (Fig.1) does not depend on k.
b) The sum $\boldsymbol{S}$ of triangular bases remains unchanged while folding, but depends on k. It becomes clear while changing the value of k, but keeping t = 0. The length of the x-axis segment, occupied by triangles, grows as k grows. It is limited by and arbitrarily close to the length of a circle. So, $\lim_{k\to\infty} S = \textit{circle length}$.

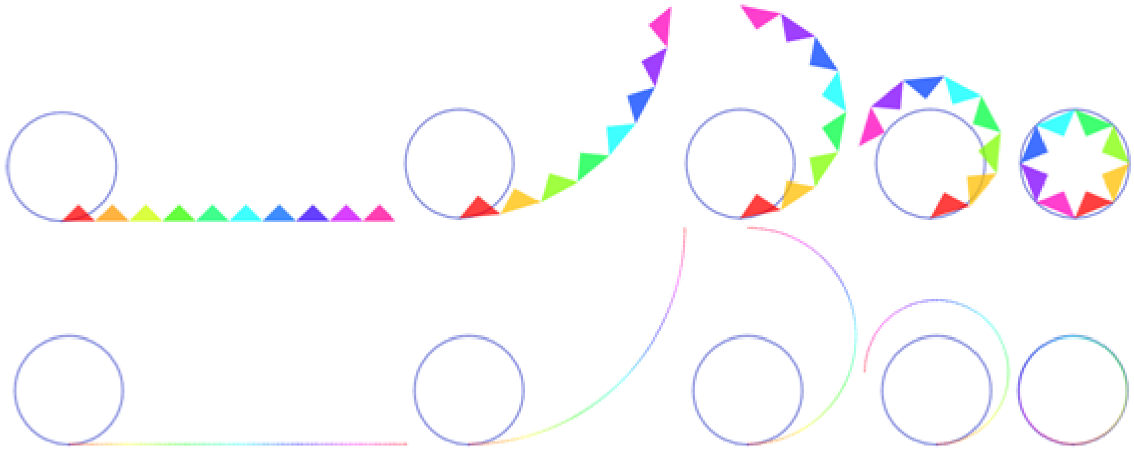


Figure 2. Upper row k = 10, lower row k = 100; from left to right: t = 0, 0.25, 0.5, 0.75, 1.

Similarly to M1, the common length of polygonal chain, formed by the lateral sides of triangles is $\sqrt{2}\,\boldsymbol{S}$ and thus it converges to $\sqrt{2} \times$ (*circle length*). However, the lower row of Fig.2 illustrates the fact that both polygonal chains are converging to the circle. So,

$$\lim_{k\to\infty} S = \lim_{k\to\infty} \sqrt{2}S = \sqrt{2}\lim_{k\to\infty} S \Rightarrow \lim_{k\to\infty} S = 0...! \quad (1)$$

Explanation of falseness of this sophism will provide a useful student activity.

One can consider the length of a curve as the length of a segment, received as result of its straightening, and the curve itself – as result of bending of this linear segment.

The following model **M3** shows the “real” bending process and helps to understand the difference and the relation between $\boldsymbol{S}$ and circle length (Fig.3)[4]. The amount of sides-segments in the model is controlled by the parameter c, and the angle of the bended arc by k.

[3] https://en.wikipedia.org/wiki/The_Rolling_Bridge

[4] In an advanced class, the discovering of *involute trajectories* of vertices during the folding process may become an interesting challenge.

Pay students' attention to the following:

1. By changing the *k* value, we are folding a segment of the horizontal *x*–axis onto the curve. The already bended portion – the circular arc - becomes rainbow-colored, as well as its preimage – a segment of *x*-axis. The red *involute* emphasizes the correspondence. Moreover, the correspondent points on arc and segment have the same color.
2. The *c* segments[5], initially reclining separately and equidistantly on light-blue horizontal lines, become connected; thus forming the resulting regular polygon.
3. The size of the horizontal light blue line and its involute does not depend on the amount of sides of the resulting folded regular polygon.

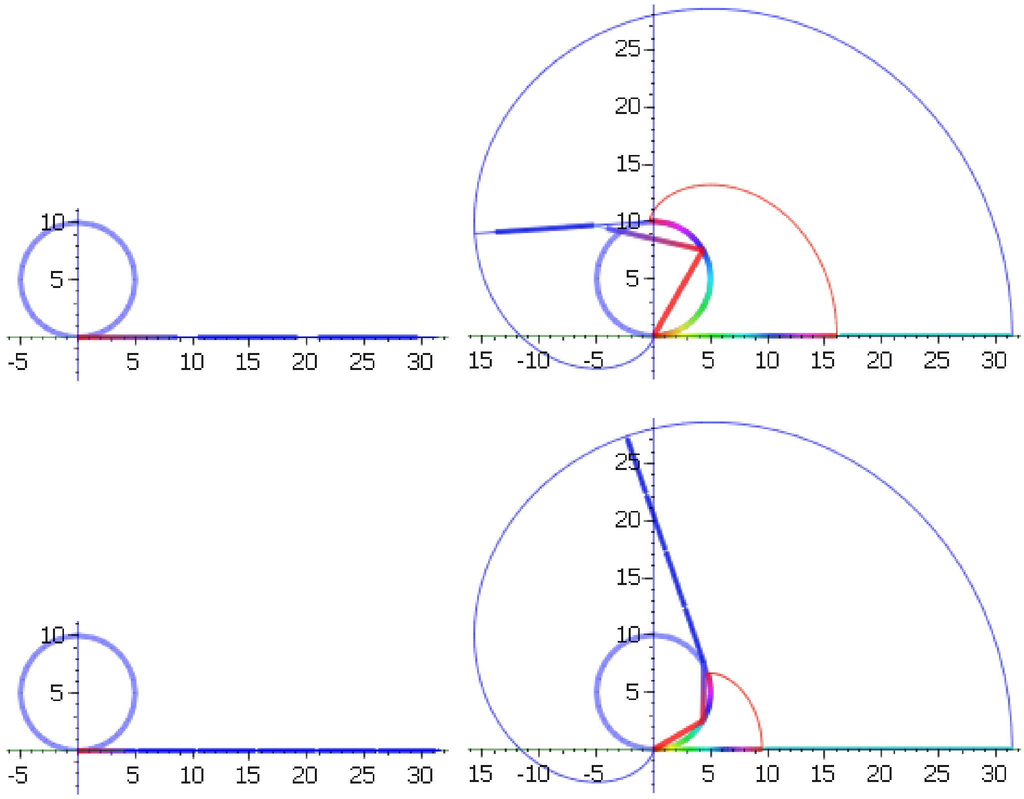


Figure 3. Upper row $c = 3$, lower row $c = 6$

One can ask students to:

1. Compare the length of the side-chord of final regular polygon with the contracted arc.

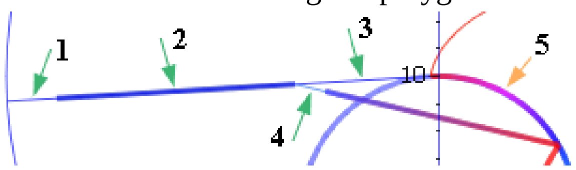


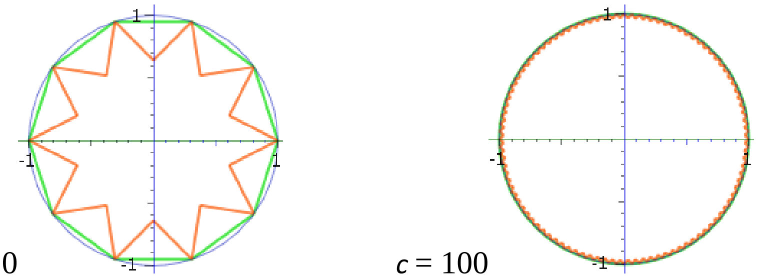
Figure 4

2. Explain the meaning, relation and comparative lengths of segments and arc, referred by arrows 1...5, pointed to their middle in Fig.4.

[5] These segments portray the bases of triangles, similar to Fig.2. We avoid drawing the backsides (wrinkles) to prevent mess of the image.

3. Explain, how this model illustrates the equality $\lim_{k\to\infty} S = \textit{circle length}$.

The next model **M4** illustrates numerically the discussed values[6]. As before, *a* here is the circle radius, and *c* – the amount of sides of the inscribed green regular polygon (Fig.5).



c = 10 *c* = 100

expression / condition	value
2*pi*a	6.2831
length(curveGreen)	6.1803
length(curveOrange)	8.7403
length(curveOrange)/length(curveGreen)	1.4142

expression / condition	value
2*pi*a	6.2831
length(curveGreen)	6.2821
length(curveOrange)	8.8842
length(curveOrange)/length(curveGreen)	1.4142

Figure 5

By increasing the value of *c* (Fig.5 shows cases of *c* = 10 and 100) and observing the corresponding changes of the values in the "*expression/condition*" view, students discover the correct tendencies.

1.3. Arbitrary curve.

1.3.1. We calculate the length of a **polygonal chain** by summing the lengths of its linear segments. Model **M5** presents a straightforward example of a polygonal chain with an infinite length. The curve (Fig.6) consists of pink vertical segments [(1/*n*, 0), (1/*n*, 1/*n*)], where *n* = 1, 2, .., *amount* - called *Verticals*, connected by blue sloped segments [(1/(*n* + 1), 0), (1/*n*, 1/*n*)], called *connectors*. The model displays two values: *length* (*Verticals*) and *length* (*connectors*). By changing the value of parameter *amount*, one can see that the total length of each one of the segments' sets, and the length of the curve (their sum), grows, but very slowly. It equals to 13.755236, when *amount* = 500.
Recognizing *harmonic series* in the *length* (*Verticals*), students get a clear understanding that both: this value and the curve length tend to infinity. Moreover, any subcurve, defined on interval [0, a], where $0 < a \leq 1$ also has an infinite length.

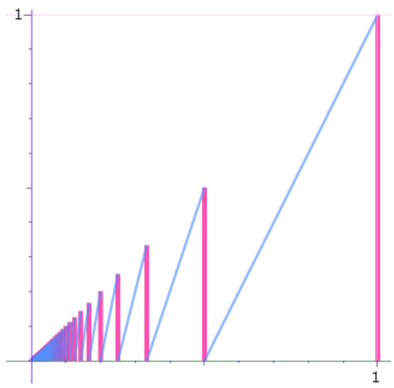


Figure 6

[6] We use *VisuMatica*'s constructs ***multiline*** (…) and ***polyline*** (…) in M3 and M4 to define sequences of isolated segments and polygonal chains.

Space filling curves provide much more interesting examples of curves with an infinite length. Consider the *Hilbert curve*, a well-known example of such a curve. Hilbert's principle is as follows. If the interval ***I*** can be continuously mapped to a square ***Q***, then after dividing ***I*** into four congruent subintervals and ***Q*** into four congruent subsquares, each subinterval can be mapped continuously to one of the subsquares. Then each subinterval, in turn, can be divided into four congruent subintervals and each subsquare into four congruent subsquares, and so on. If this goes on indefinitely, then ***Q*** and ***I*** break up into 2^{2n} congruent copies for n = 1, 2, 3. Hilbert showed that it is possible to arrange subsquares so that the adjacent subintervals would correspond to adjacent subsquares with the common side, and so that the inclusion relation is preserved, i.e. if the square corresponds to an interval, then its subsquares correspond to the subintervals of this interval [8].

Model **M6** implements this construction. Fig.7 displays the cited process for the few starting steps (1, 2, 3, 5, and 10). The spectrum-rainbow colored horizontal segment models interval ***I***, and the unit square – the square ***Q***. The subdivision squares are presented by the gridlines. Color codes expresses the mapping correspondence: the point with coordinate ***c*** on interval ***I*** and its image – point (x, y) in ***Q*** have the same color.

Considering **step-curves** (polygonal chains) as approximations of the Hilbert curve[7] one can see that their length - *length* (*Hilbert*) - grows very quickly[8]. However, it remains not clear whether this growth is limited or not, and thus, whether the Hilbert curve has an infinite length.

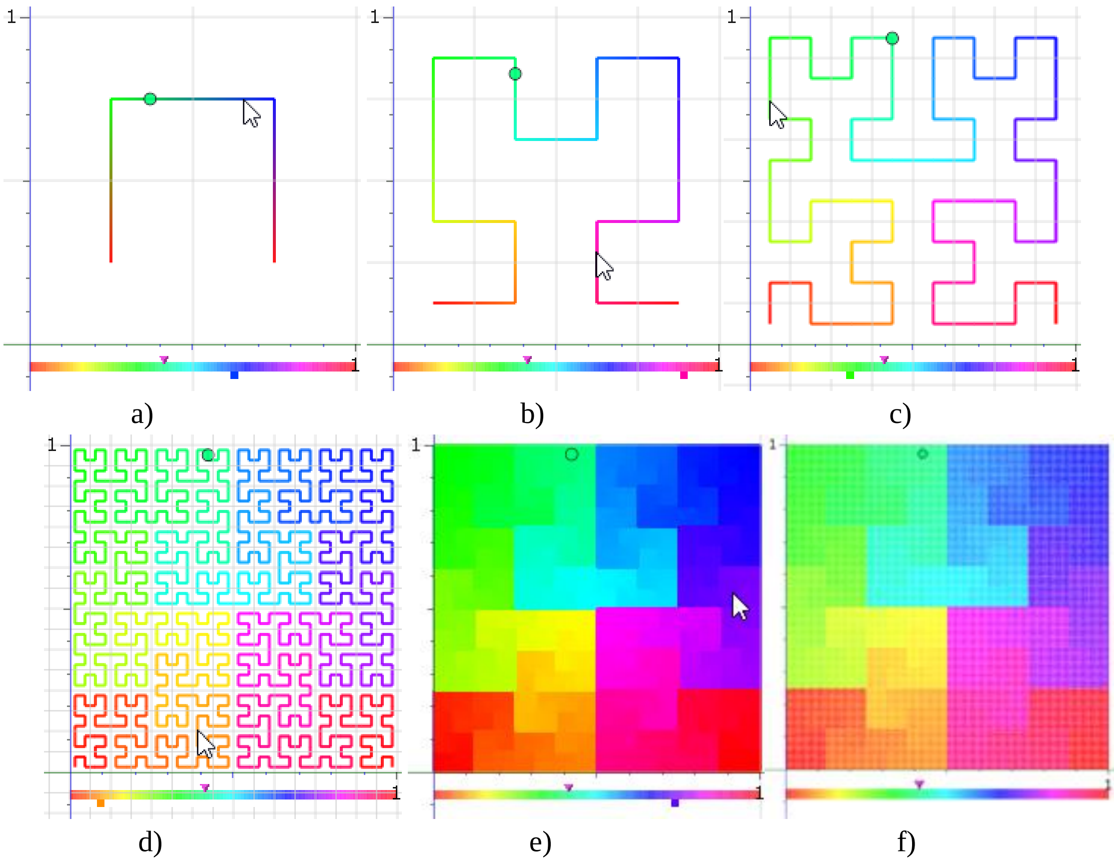


Figure 7. Steps: a) 1, b) 2, c) 3, d) 5, e)-f) 10

[7] Defined as an overloaded *polyline* that depends on steps.

[8] *VisuMatica* shows the following values: 1.5 (k = 1), 31.96875 (k = 5), 1023.999023 (k = 10) etc.

On the other hand, it is clear that the Hilbert curve fills the whole ***Q***. It *looks as* even the "curve" in step 10 (Fig.7 e) completely fills out the square. Nevertheless, every *rectifiable curve is a zero set*, that is, *it can be enclosed in a polygonal figure of an arbitrarily small area*. Considering as such a figure *F* the set of all points distanced from the points of the curve by a distance not exceeding ε, the length of the curve can be understood as $\lim_{\varepsilon\to 0}\frac{S(F)}{\varepsilon}$, where $S(F)$ is the area of the figure *F*. The Hilbert curve has $S(F) \geq 1$. Hence, in *sense of such definition, its length is infinite*.

Positions of both the small triangle above the colored interval and the circle on the approximation polygonal chain in Fig.7 depend on parameter $c \in [0, 1]$, and illustrate the correspondence $f(c) = (x, y)$. The triangle points to a light green zone of a rainbow while the inner disk of the circle covers the accordant and thus light green part of the approximation chain. A small change of the *c* value leads to also small changes of positions and colors of triangle and the correspondent circle.

The second pair of objects - the mouse and the square under the rainbow interval - illustrate the reverse correspondence. When the mouse points to a certain point of the curve, the square moves to the preimage of this point - a point with the same color on the interval[9]. We draw the students' attention to the fact that smooth movement of the mouse along the curve leads to a smooth displacement of the square only in cases of a) - d). In the case of e) sharp jumps in the position of the square clearly are observed at small displacements of the mouse, when it crosses the boundaries of zones that differ sharply in color. The explanation of this phenomenon is based on a comparison of the images of steps a) -e).

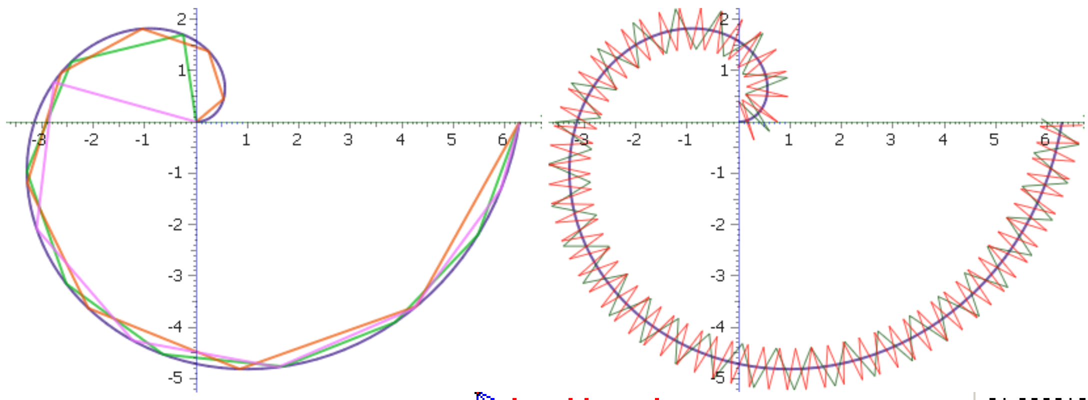
Students discover two interesting facts: (a) although ALL steps represent a bijective correspondence[10], the Hilbert curve itself is a result of only a surjective mapping of an interval onto a square, (b) and this mapping is continuous.

1.3.2. Let us approximate a curve by a polygonal chain, defined by a finite number of points on the curve. The length of the approximation is the sum of lengths of linear segments. Increasing the number of sides while decreasing length will lead to better approximations.

Curves, for which there exists $L > 0$ - an upper bound of lengths of such approximations, are called *rectifiable*. Their *length* is defined as the number *L*. Fig.8 presents model **M7** that illustrates this definition.

[9] *VisuMatica* shows the value of expression *pointedColor*(*mouse*), which is $c \in [0, 1]$.

[10] Fig.7 f) presents step 10 by polygonal chain with line width of one pixel.



c		58
d		2.87
n		9
divergence		0.4

length(curve)	21.269616
length(polylineEqArcs)	20.270332
length(polylineEqParam)	20.702802
length(polylineEqSides)	18.582578
length(polylineZigZag)	50.852422
length(polylineWild)	102.147235

Figure 8

The model includes the dark blue spiral "*curve*" defined as $r = \Theta$, $\Theta \in [0, 2\pi]$ and five polygonal chains:

a. Light green *polylineEqArcs*, defined as *subdivision(curve, n, 1)*,
with *n* segments, connecting arcs with equal lengths.

b. Orange *polylineEqParam*, defined as *subdivision(curve, n, 0)*,
with *n* segments, connecting arcs with equal parameter Θ steps.

c. Pink *polylineEqSides*, defined as *subdivision*(*curve*, *d*),
with segments of equal length *d*, *except maybe the last one*.

d. Dark green *polylineZigZag*, defined as *polyline(curve, divergence, c, 1)*,
with *c* segments, whose ends are *divergence* far from the curve.

e. Red *polylineWild*, defined as *polyline(curve, divergence, 50/divergence, 1)*,
with *50/divergence* segments, whose ends are *divergence* far from the curve.

The left image in Fig.8 shows polygonal chains a), b), c), and the right one – chains d), e). The bottom-left table presents the proper values of parameters, and the values of lengths expressions are presented in the bottom-right one[11].
Students pay attention that:

- The polygonal chains a) – c) present different approximations of the curve.
  Their length increases but is limited by the curve length, when the amount of their segments is growing (by increasing the value of *n* parameter (polylines a), b) or decreasing the value of *d* parameter for polyline c)).

[11] It is up to educator to include expression *length(curve)* and/or *polylineEqArcs* in the model.

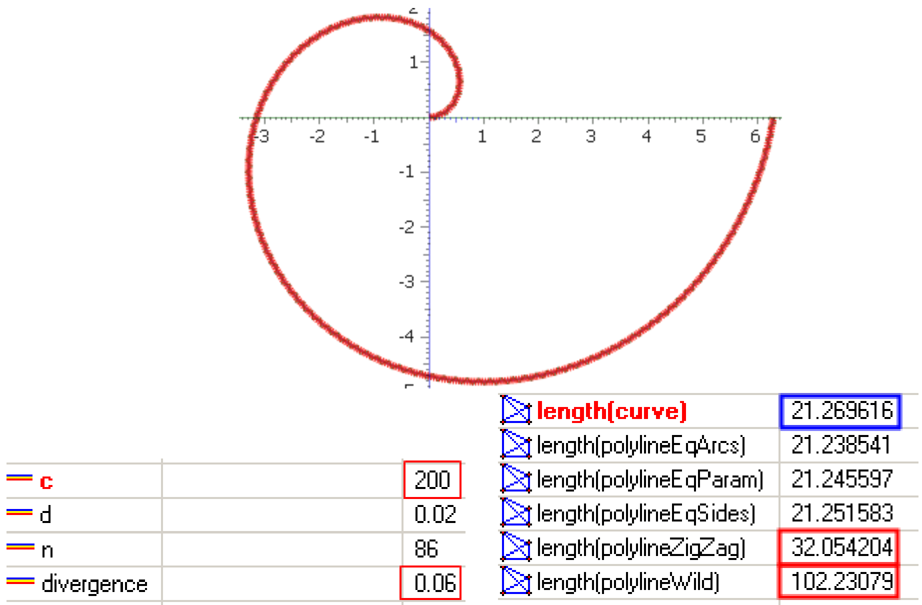


Figure 9

- The length of both polygonal chains d) and e) is bigger than the curve length.
  Both these polygonal chains are approaching to the curve when decreasing the value of parameter *divergence*.
- The length of polyline d) decreases and approaches the curve length, when decreasing the value of *divergence* but increases back if *c* increases (Fig.9).
- The length of polyline e) remains almost unchangeable, when decreasing the value of parameter *divergence* (Fig.9).

Analyzing these observations, students find an analogy with the case (**1.2**) of the circle length and come to the conclusion that only a certain class of polygonal curves can lead to the concept of curve length. An approximation to the curve alone is not enough[12].

2. Surface Area

By analogy with a circle, we construct an approximation of a *sphere* by polyhedra (model **M8**). Parameter *subdiv* presents the stage of approximation. As an initial approximation, we take an icosahedron with vertices on the sphere (*subdiv* = 0). Each step of approximation is a polyhedron obtained from the previous one in the following way:

a. The middle points of the edges are projected centrally onto the surface of the sphere.
b. The set of points, which consists of points-projections and existing vertices, is the set of vertices of the new polyhedron. Its edges join pairs of nearest vertices, and its faces are regular triangles with sides-edges.

[12] As always, it is possible and useful to redefine the original curve, making sure that all "broken" and discussed properties are preserved.

The upper row on Fig.10 confirms the fact that these polyhedra[13] do approach the surface of the sphere.

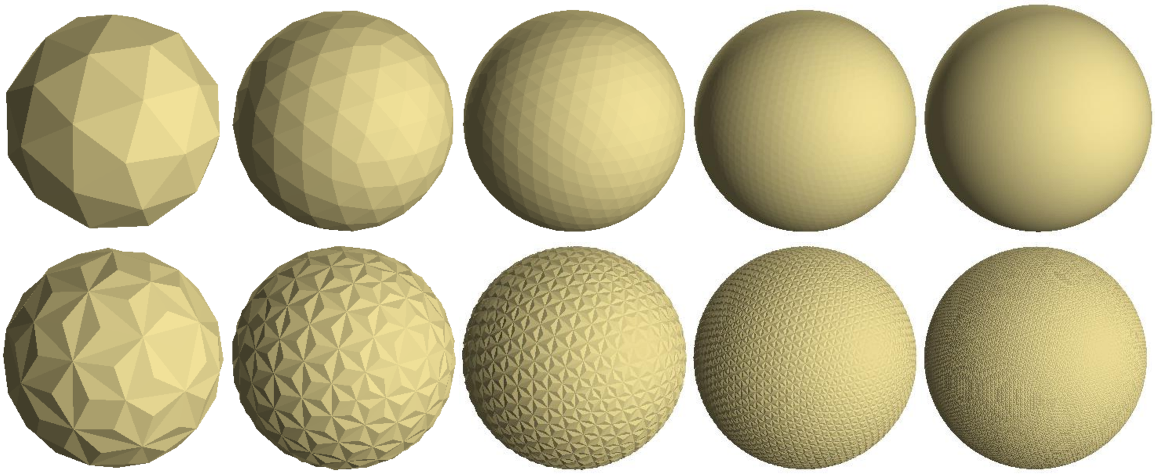


Figure 10. From left to right *subdiv* = 1, 2, 3, 4, 5.

Let us modify these polyhedra by erecting right tetrahedrons on their triangular facets.
The second parameter – *thorn* – in M8 defines the angle of a slope of the pyramid's apothem to the base plane.

The lower row on Fig.10 shows the polyhedra corresponding to *thorn* = π/10. These polyhedra approach the surface of the sphere too. Denoting area by letter ***S*** we have

$$S_{lateral\ surface\ of\ pyramid} = \frac{S_{base\ of\ pyramid}}{\cos(thorn)} \Rightarrow S_{polyhedron\ with\ spikes} = \frac{S_{polyhedron\ without\ spikes}}{\cos(thorn)}$$

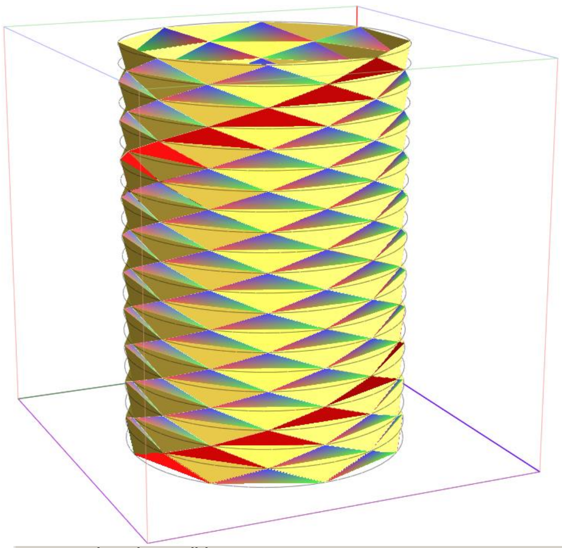
Therefore, the surface area of the polyhedron with spikes is $\dfrac{1}{\cos(thorn)}$ times bigger than without them.

Students meet here a situation that similar to (1) in the section 1.2…

This was expected: not all vertices of the polyhedron lie on the surface of the sphere. But it is not enough. An attempt to determine the surface area by analogy with the length of a curve as the limit of the area of an inscribed polyhedron with all vertices lying on the surface and faces, whose area tends to zero, turns out to be unsuccessful.

A famous counterexample was constructed by Hermann Schwarz in 1892 and got his name "*Schwarz boots*" because of its similarity with the wrinkled boots. Fig.11 shows model **M9** of *Schwarz boots* – a polyhedral surface inscribed into cylinder with radius *a*, height *zMax* – *zMin* of the viewing volume and axe *z* as its axis, standing on the *x*O*y* plane (*zMin* = 0).

[13] With vertices on the surface of the sphere and diminishing faces.



name	expression	value
a		7
b		1
c		0
d		0
n		8
k		21
h	(zmax-zmin)/k	0.952380952381
tik	n+k+3	32

expression / condition	value
2*pi*a*(zMax-zMin)	879.645943005142
a*sin(pi/n)*sqrt(a^2*(1-cos(pi/n))^2+((zMax-zMin)/k)^2)	2.923376328134
2*n*k*a*sin(pi/n)*sqrt(a^2*(1-cos(pi/n))^2+((zMax-zMin)/k)^2)	982.254446253052

Figure 11

Fig.12 shows process of construction of the polyhedral surface, controlled by parameter *tik* (first nine steps a) - b) correspond to *tik* = 0...9). Here emphasized is the construction mechanism of the initial (red) triangular facet (d); construction of the complementary yellow facet as symmetric (e) and then rotated by an angle π/n (f). The section fulfilled by rotation of this pair of triangles is given by an angle $2\pi/n$ (g, h).

Each following section is rotated relative to the previous one by an angle π/n. This became visible by the onward movement of the red initial faces (Fig.12 i) and Fig.11).

Thus, the polyhedral surface consists of congruent triangular facets. Each such facet has[14]

$$base = 2a\sin\frac{\pi}{n}, \qquad height = \sqrt{\left(a - a\cos\frac{\pi}{n}\right)^2 + \left(\frac{H}{k}\right)^2}$$

Its area is $S_\Delta = a\sin\frac{\pi}{n}\sqrt{a^2\left(1-\cos\frac{\pi}{n}\right)^2 + \left(\frac{H}{k}\right)^2}$.

[14] In model M9 we use expression *zMax – zMin* instead of *H* to construct cylinder of a maximal height, allowed by the viewing volume.

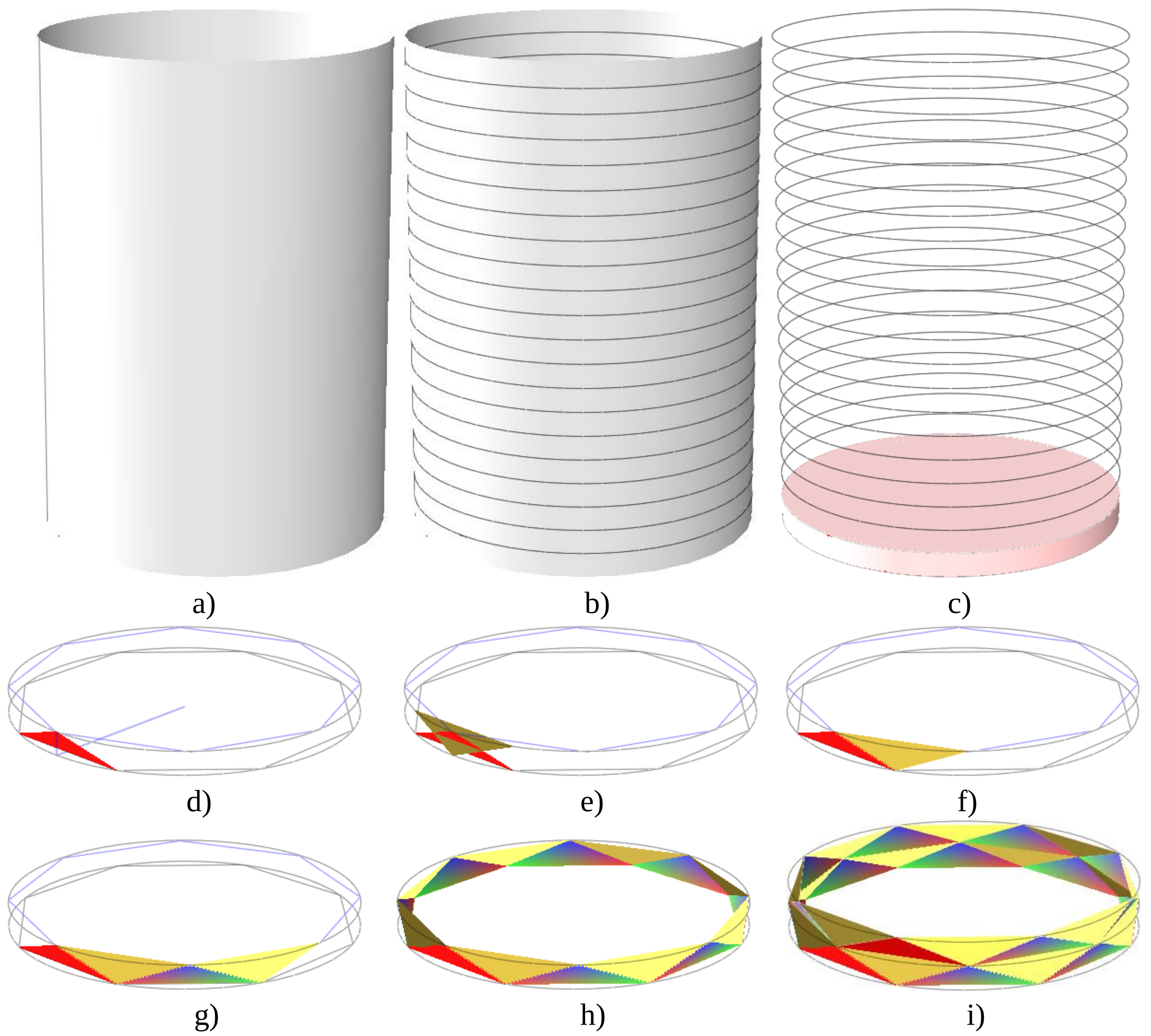


Figure 12

Therefore, the surface area is $\sum_{k,n} = 2knS_{\Delta} = 2an\sin\frac{\pi}{n}\sqrt{a^2k^2\left(1-\cos\frac{\pi}{n}\right)^2+H^2}$.

If *m* and *n* are growing infinitely and independently, then the sizes of triangles tend to zero but the area $\sum_{k,n}$ does not have a limit.

If *m* and *n* are growing in such a way that $\lim\frac{k}{n^2} = \boldsymbol{b}$ then, taking into account that

$\lim\left(n\sin\frac{\pi}{n}\right) = \pi$, and $\lim k\left[\left(1-\cos\frac{\pi}{n}\right)\right] = \lim\left(k2\sin^2\frac{\pi}{2n}\right) = \lim\left(\frac{\pi^2}{2}\frac{k}{n^2}\right) = \frac{\pi^2}{2}b$, we have

$$\lim\sum_{k,n} = 2\pi a\sqrt{\frac{\pi^4a^2}{4}b^2+H^2} .$$

This limit depends on $\boldsymbol{b}$: for $\boldsymbol{b} = 0$, i.e. $n\to\infty$ while k is fixed, it equals to $2\pi aH$ in accordance with the well-known formula. In other cases, it is larger and can be infinity or equal to an arbitrary number. Thus, the *Schwarz boots* explains the inapplicability of the approach to determining the surface ***S*** area through the sum of the areas of the contracting faces of a polyhedral surface with vertices on ***S***.

Students explore these facts by changing parameters n and k. In particular, they redefine k by bn^2, set the value of b, and change the value of n, paying attention to the changing values of expressions in the *expression/condition* window.

3. Continuity, Differentiability and Extrema

In 1872, Karl Weierstrass first published an example of everywhere continuous but nowhere differentiable function. This counterexample has ruined the accepted opinion that every continuous function is differentiable except on a set of isolated points.

Weierstrass Function (further: ***WF***) is defined as

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \qquad (2)$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + 3/2\pi$. This significant fact is depicted on a post stamp, dedicated to Karl Weierstrass (Fig.13).

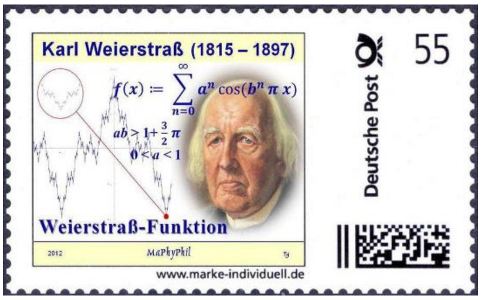
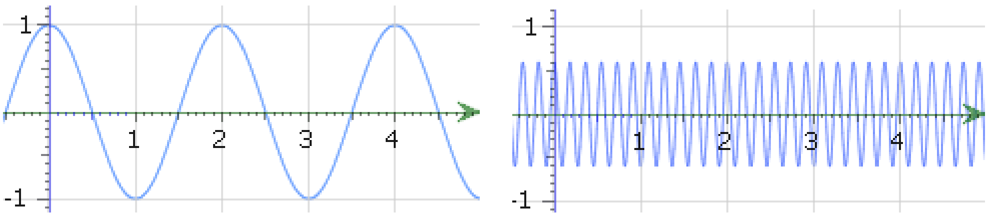


Figure 13

Model **M10** presents a graph of *WF*. Fig.14 shows the idea of its construction, based on initial function f_1: $y = \cos x$. Each summand in (2) is a function $y = a^n f_1(b^n x\pi)$. Fig.14 a) display its graph, when $n = 0$, and b), - when $n = 1$. Here $a = 0.6$, and $b = 11$. Fig.14 c) shows graph of their sum $y = \sum_{n=0}^{1} a^n \cos(b^n \pi x)$ together with graph of $f(x) = \sum_{n=0}^{0} a^n \cos(b^n \pi x)$. Generally, M10 shows both graphs: $y = \sum_{n=0}^{k-1} a^n \cos(b^n \pi x)$ in brown and $y = \sum_{n=0}^{k} a^n \cos(b^n \pi x)$ in green[15], depending on the **finite** value of parameter k as amount of summands in (2).



a) b)

[15] In *VisuMatica* this expressions supported by the following syntax: *y=Sigma*($a^n f_1(b^n \pi x)$,*n*,*0*,*k*).

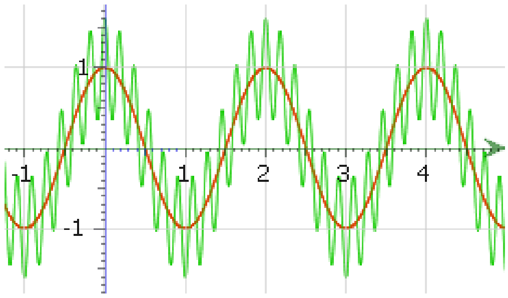


Figure 14

Exporation of this model includes two types of activities:

1. Increasing the value of parameter *k* (Fig.15 1st row).
2. Zooming to view definite regions of the graph (Fig.15 2nd row, both images with $k = 10$).

As result, students discover the self-similarity of the graph, and after deep zooming - its similarity to the graph with low-valued *k* parameter[16], e.g. Fig.14 c).

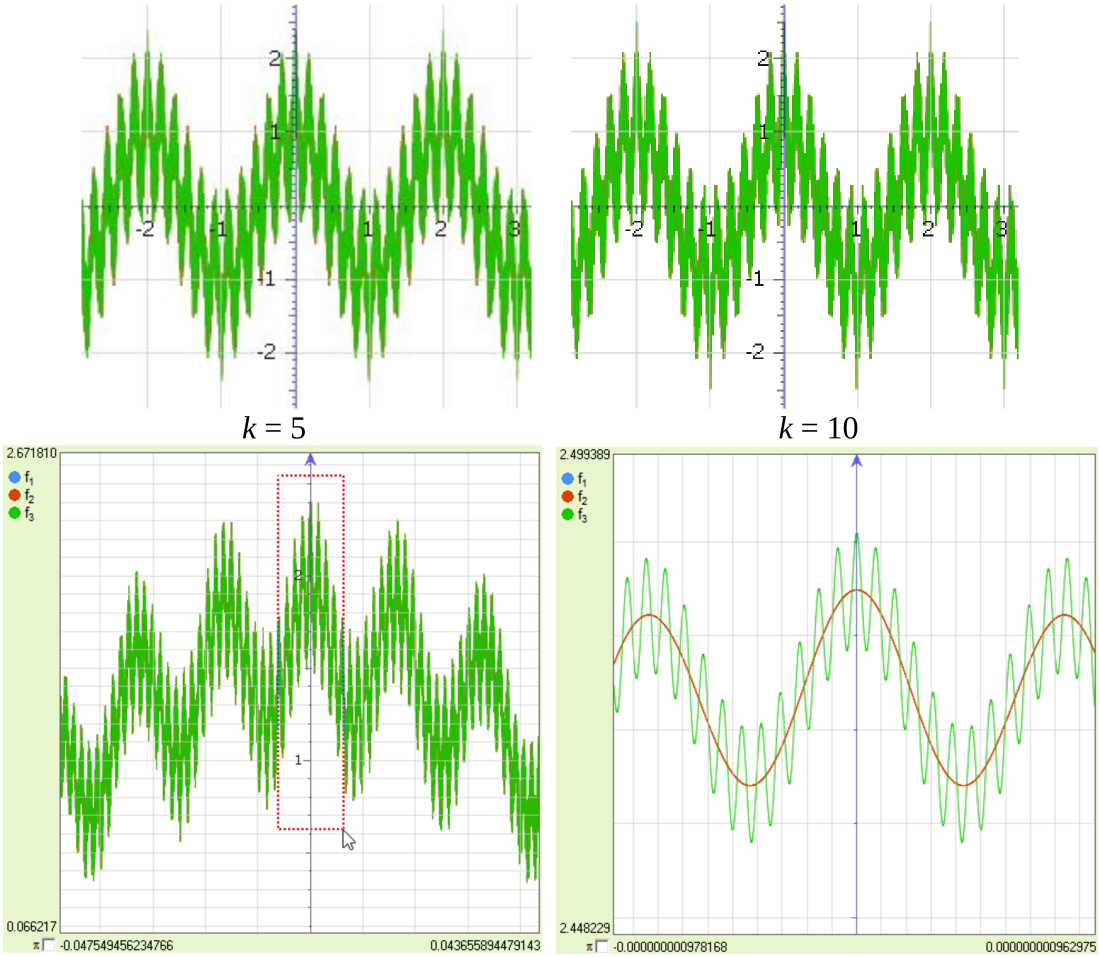


Figure 15

[16] While proper zooming pay students' attention to the viewed *x*- and *y*-interval (Fig.15 right).

The continuity of $f_k(x)=\sum_{n=0}^{k} a^n \cos(b^n \pi x)$ becomes visible. One can prove the continuity of its limit – the ***WF***, using the concept of *uniform convergence* of a sequence of functions[17], *Weierstrass M-test*[18], *and* the *uniform limit theorem*[19]. To do this, it is sufficient to note that
0 < *a* < 1. Hence, $a^n < 1$, being the amplitude of the summands, forms a decreasing geometric progression. $|\sum_{n=0}^{k} a^n \cos(b^n \pi x)| \le \sum_{n=0}^{k} a^n$ and $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty$. Therefore, ***WF*** *converges uniformly* on *R*. Consequently, it is continuous.

The role of the parameter *b* in the argument's expression of $f_k(x)$ is clear and confirmed by varying of its value: the coefficient *bn*π makes the period of the function of the leading term equal to $\frac{2}{b^n}$ - the tightness of the graph's bursts. However, the meaning of condition $ab > 1+\frac{3}{2}\pi$ is difficult to elucidate experimentally. Observation of model's behavior, while varying values of both parameters *a* and *b*, allows only to note that there really exists a certain relationship between them, which determines the nature and steepness of the bursts right up to their vanishing.

Unfortunately, the smooth zoomed images - like in Fig.15 (right in the 2nd row) – disturb to "see" the lack of ***WF*** differentiability. The formal proof includes rather cumbersome and non-trivial computations and argumentations[20] [10], [11]. We can make the bursts of graph sharp by replacing the initial function $y = \cos x$, by another one, whose graph will be similar to the graph, shown in Fig.1, and extended to the whole *R*.

The geometric construction with *polyline*, used in M1, is unacceptable since the number of segments in the polyline must be finite. Moreover, it is does not compatible with *sigma* expression. Finding the right function becomes an interesting challenge for students.
Consider $y = |\{x\text{-}0.5\}\text{-}0.5|$ as an example of such a function[21]. This function is especially fascinating: it provides an example of a transformation of a noncontinious function with infinite amount of isolated gaps into a continuous one.

VisuMatica includes a "Steps" mechanism, which allows one to trace the process of its formation and application. Model **M11** includes the following functions:

- $f_1(x)=\{x\}$,

---

[17] We say that sequence of functions $\{f_n\}$ defined on a set *E* <u>*converges uniformly*</u> to function *f* on *E* if $\forall \varepsilon > 0, \exists N(\varepsilon), \forall x \in E, \forall n > N(\varepsilon) : | f_n(x) - f(x) | < \varepsilon$

[18] *Weierstrass M-test.* If $\{f_n\}$ is a sequence of functions defined on a set *E*, that there is a sequence $\{M_n\}$ satisfying $\forall n \ge 1, \forall x \in E : | f_n(x) | \le M_n \wedge \sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly* on *E*.

[19] The *uniform limit theorem*: The uniform limit of any sequence of continuous functions is continuous.

[20] Including condition $ab > 1+\frac{3}{2}\pi$

[21] We denote the fractional part of a real number *x* by $\{x\}$

- $f_2(x)=f_1(x\text{-}0.5)$,
- $f_3(x)=f_2(x)\text{-}0.5$,
- $f_4(x)=|f_3(x)|$,
- $f_5(x)=sigma(a^n f_4(b^n x),n,0,k\text{-}1)$,
- $f_6(x)=sigma(a^n f_4(b^n x),n,0,k)$, and parameters a, b, and k.

Initially, graphs of all the functions except $f_1(x)$ are invisible; $a = 0.51$, $b = 4$, and $k = 2$. Fig.16 displays the "*Steps*" dialog box. Its list includes names of graphing objects, e.g. functions, and variables. Step-by-step processing of the list:

- makes objects of the active step (emphasized in red) visible and hides the rest.
- Assignes definite value to parameters and respectively redraws the current scene. In case of loops (see the line "k:=3..7") the assignment executes sequentially.

Fig.17 shows results of steps execution: a) – f) steps 1 – 6 respectively, g) – step 7 with k = 3, h) – step 7 with k = 7. While zooming students discover the self-similarity of the graphs like in Fig.15.

Consider $f_6(x)=sigma(a^n f_4(b^n x),n,0,k)$. It is continuous as a sum of a finite amount of continuous summands (Fig.18). As in case of *WF*, if we require $0 < a < 1$. Then $a^n < 1$. Fig.17 d) illustrates the fact that $|f_4(x)| = |\{x\text{-}0.5\}\text{-}0.5| \le 0.5$.

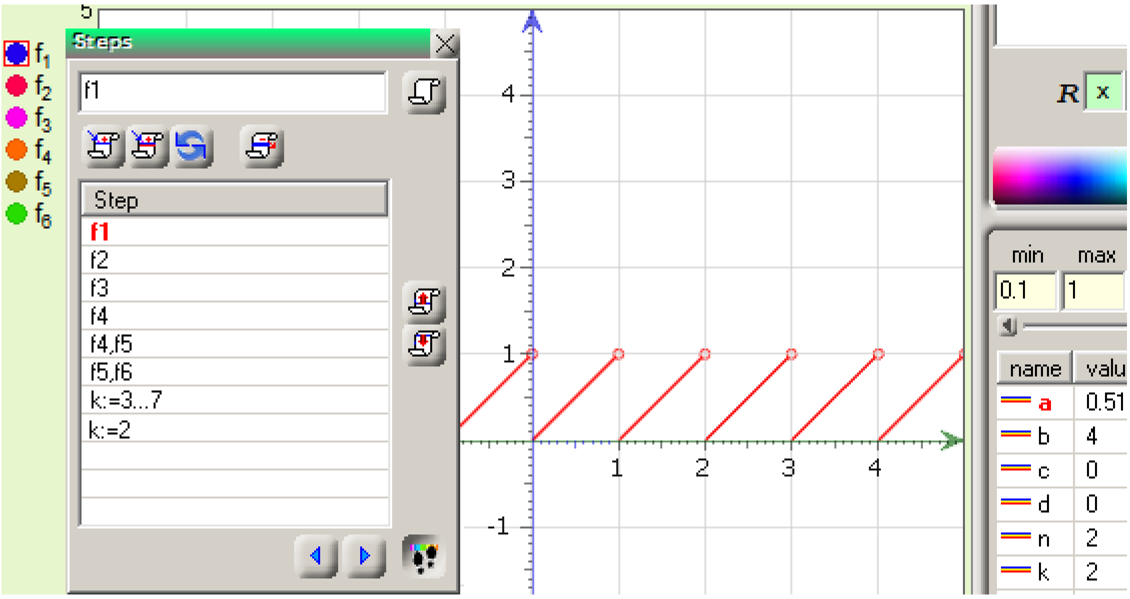


Figure 16

We can see that $|f_6(x)| = |\sum_{n=0}^{k} a^n |\{b^n x - 0.5\} - 0.5|| \le \sum_{n=0}^{k} a^n$ and $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty$. Therefore, the sequence $f_6(x)$ *converges uniformly* on R. Consequently, its limit $\boldsymbol{F}$ is a continuous function. As in case of *WF*, parameter a defines the amplitude of summands, and b – their period. We confine ourselves to considering only even b to ease the farther observations, especially, the visual ones.

As a result of studying the shapes of summands, students note that they include an infinite amount of regularly distributed sharp minima and maxima, which converges to a countable dense subset of R. In these points summands do not have derivative.

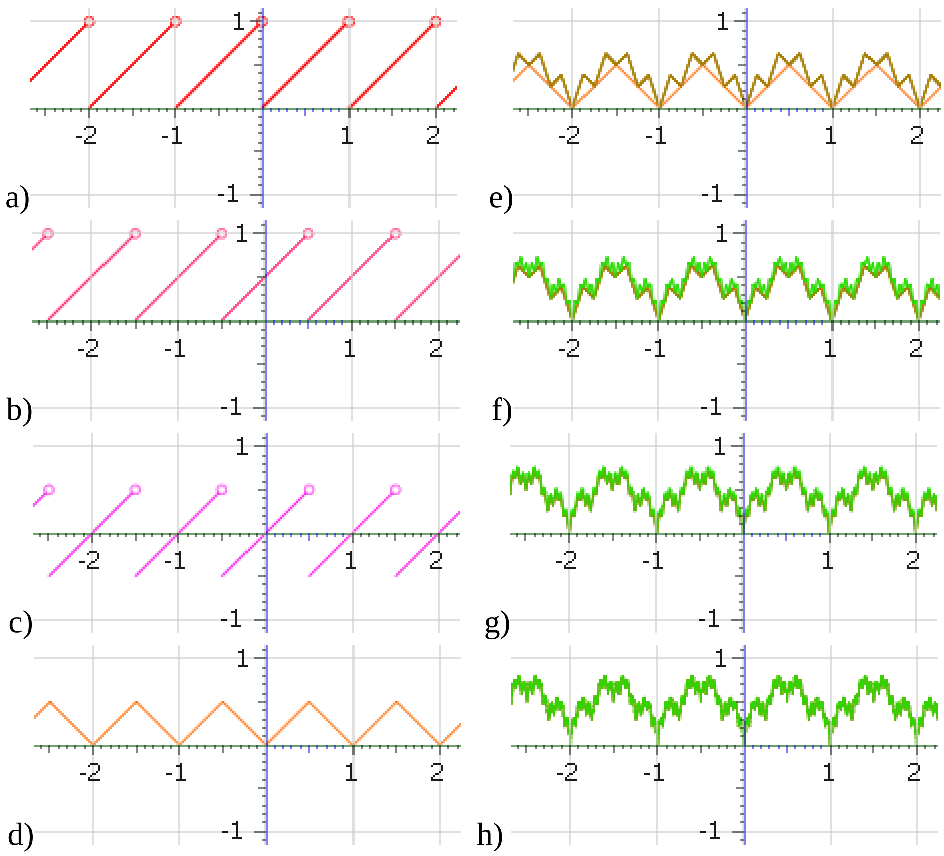


Figure 17

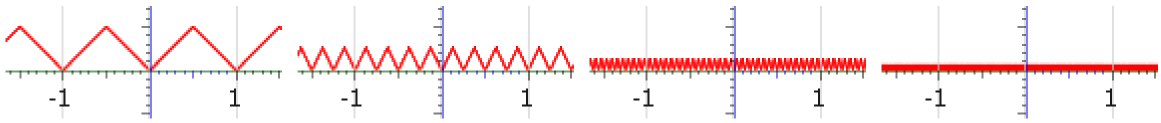


Figure 18. Graph of summand $a^n f_4(b^n x)$, if a=0.51, b = 4, n = 0, 1, 2, 3 from left to right

Fig.19 illustrates the process of summation. Students pay attention to the location of break points of the last summand $y=a^k f_4(b^k x)$ (the magenta line) in relation to the previous sum (the brown line), and the sum $f_6(x)$ (the green line). They remain in place in all subsequent steps.

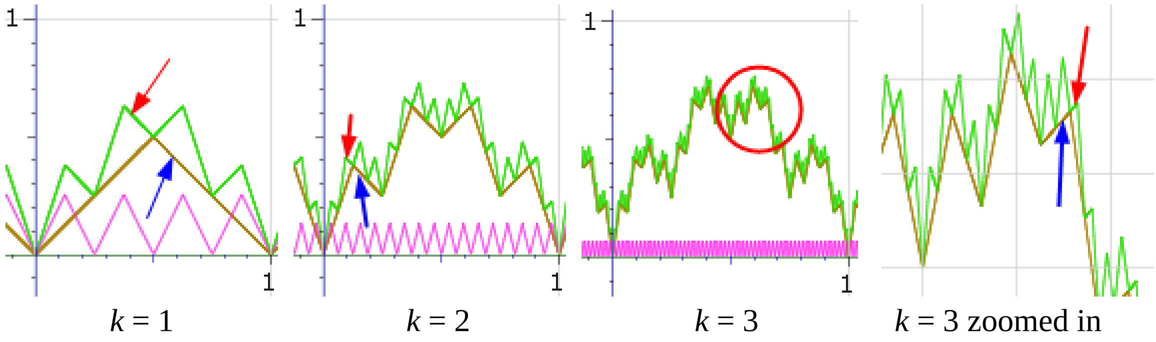


Figure 19. a=0.51, b = 4

We want to ensure the lowest slope's modulus of the sum of k summands (red arrows in Fig.19) to be bigger than the lowest slope's modulus of the sum of k -1 summands (blue arrows in Fig.19). It will guarantee preservation of minima and maxima of the last summand (k) in the result of addition[22].

The main experiment with this model is the search for a relation between the parameters ***a*** and ***b***, if any, that ensures the preservation of extrema. Students are invited to:

1. Successively double the value of parameter ***b***, and then
2. By varying parameter ***a*** and, if necessary, by zooming[23], to determine its *critical value* ***a'*** (if exists at all) - i.e. a value that for ***a*** **<** ***a'*** the minimal slope of the sum decreases, while for ***a*** **>** ***a'*** it increases.
3. Check the independence of ***a'*** from the number ***k*** of terms.

Making sure that ***a'*** really does not depend on ***k***, students discover the following pairs: b=4, a'=0.5; b=8, a'=0.25; b=16, a'=0.125 and easily notice that $a'b = 2$, and, therefore, *the condition of increasing* is ***ab*** **>** ***2***.

They check this guess by assigning different even values to ***b*** and setting $a = \frac{2}{b}$. The graph of $f_5(x)$ and $f_6(x)$ receive a proper shape: *the lowest slopes coincide*. This fact follows from the collinearity of the corresponding segments: green is a continuation of brown[24] as it emphasized in Fig.19 by arrows.

As a result, it can be affirmed that the set of sharp breaks of the function ***F*** is everywhere dense on R, and therefore the function is not differentiable at these points. In an arbitrarily small neighborhood of *any* $x \in R$ there are increasing segments with a slope of at least 1, i.e. $\frac{\Delta y}{\Delta b} > 1$, and decreasing, with a slope of not more than -1, i.e. $\frac{\Delta y}{\Delta b} < -1$. Thus, the $\lim_{\Delta b \to 0} \frac{\Delta y}{\Delta b}$ does not exist, and hence the function ***F*** *is nowhere differentiable*. Students pay attention that the function F*, being continuous, has a dense set of extrema*. This would be impossible if F would be differentiable.

Replacing $y = \cos x$ by $y = 4|\{x/(2\pi)\}-0.5|-1$ as $f_1(x)$ in the definition of *WF*, we transform the *WF* into a representative of the function F (Fig.20: left - *WF*, right - F).

Short history

It is worth noting an interesting connection between authors of the mentioned counterexamples. Here one can not do without another outstanding mathematician - Giuseppe Peano. It was he who discovered the first example of a space-filling curve (1890). David Hilbert proposed his curve in 1891 as a variant of Peano curve. In 1882 Peano independently from Schwarz also discovered the Cylinder Area Paradox. It was the his first result in Calculus. When Peano told this to his teacher Angelo Genocci he informed him that already in 1880 he received a letter from Hermann Schwarz describing this discovery. Schwarz had officially published his counterexample only in 1890.[25]

[22] The arguments of the extrema of the summand k include all the previous ones.

[23] *VisuMatica* supports user-friendly proportional zooming around any point of the viewport.

[24] Odd ***b*** also fits. The segments just become parallel instead of be collinear.

[25] One more link: *Karl Weierstrass was the teacher and PhD advisor of Hermann Schwarz.*

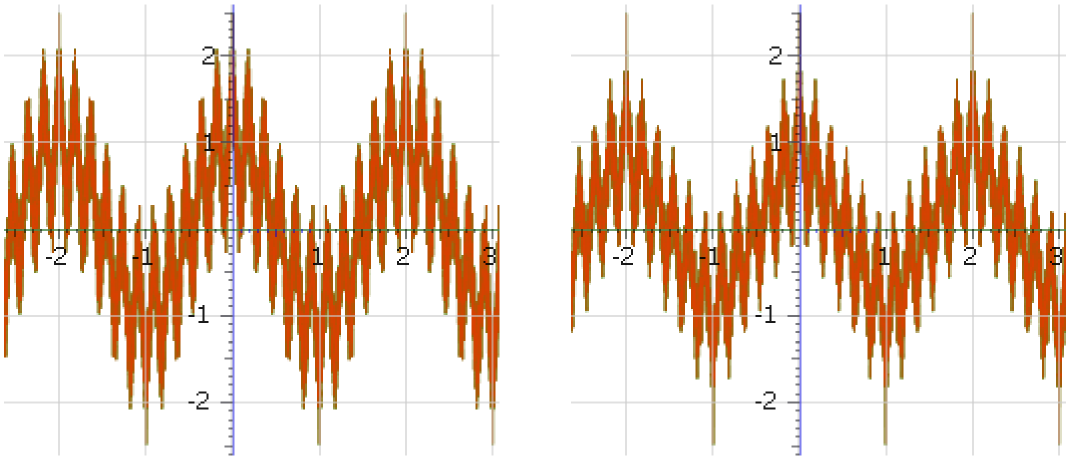


Figure 20

Conclusions

The above examples show a wide range of possibilities and helpfulness of studying various mathematical topics with application of counterexamples. At the same time a proper educational software allows to visualize and explore this contents and ensure its active assimilation.

Supplementary Electronic Materials

Videos with animations: https://sites.google.com/view/counterexamples-in-mathematics

VisuMatica in a configuration that supports the above modeling is under construction.

References:

[1] Gelbaum Bernard R., Olmsted John M. H., *Counterexamples in Analysis*. Dover Publications Inc., 2003

[2] Bourchtein A., Bourchtein L., *Counterexamples on Uniform Convergence: Sequences, Series, Functions, and Integrals*, John Wiley & Sons, 2017

[3] Bourchtein A., Bourchtein L., *CounterExamples: From Elementary Calculus to the Beginnings of Analysis.* CRC Press, 2015

[4] Jarnicki Marek, Pflug Peter, *Continuous Nowhere Differentiable Functions. The Monsters of Analysis*, Springer, 2015

[5] Klymchuk Sergiy, *Counterexamples in calculus.* MAA, 2010

[6] Singh A. N., *The Theory and Construction of Non-Differentiable Functions.* Lucknow: Newul Kishore Press, 1935.

[7] Rajwade A.R., Bhandari A.K., *Surprises and Counterexamples in Real Function Theory*. HBA, 2007

[8] Sagan Hans, *Space Filling Curves*. Springer, 1994

[9] Boltiansky V.G., *Curve Length and Surface Area* //The Encyclopedia of Elementary Mathematics, Vol.5, Nauka, 1966

[10] Weierstrass K., *On Continuous Functions of a Real Argument that do not have a Well-defined Differential Quotient.* In: Classics on Fractals/Ed. Gerald A. Edgar Westview Press 2004, pp. 3-9

[11] Hardy G.H., *Weierstrass non-differentiable function.* In: Transactions of the American Mathematical Society, Vol. 17, No. 3 (Jul., 1916), pp. 301-325

[12] Bader Michael, *Space-filling curves. An introduction with applications in scientific computing*. Springer, 2013

[13] Cesari L., *Surface Area.* Princeton U. Press, 1956